Brézin-Zee dynamical correlator: An S-matrix Brownian motion approach

A. M. S. Macêdo
Laboratório de Física Teórica e Computacional, Universidade Federal de Pernambuco, 50670-901 Recife, PE, Brazil
(Received 16 October 1996)

We study the smoothed dynamical density-density correlator of the eigenvalues of random matrices taken from certain types of ensembles. This quantity has recently been derived by Brézin and Zee for random Hermitian matrices. Our approach is based on an S-matrix Brownian motion model. It provides exact results that extend Brézin and Zee's calculation to a larger class of matrices and is also technically much simpler. [S1063-651X(97)10902-3]

PACS number(s): 02.50.−r, 05.40.+j, 05.45.+b

I. INTRODUCTION

Random-matrix theory (RMT) is perhaps one of the most powerful nonperturbative techniques described in past and recent literature [1]. Its range of application is impressive and includes essentially all energy scales in physics, going from condensed matter to nuclear reaction theory and elementary particle physics. The two most important predictions of RMT are that properties concerning the full eigenvalue support are largely nonuniversal (as, e.g., the average eigenvalue density), whereas eigenvalue statistics on the scale of the mean level spacing are universal and depend solely on generic symmetries of the system (such as the presence or absence of time-reversal invariance). It is currently widely acknowledged that a RMT approach is the most appropriate one when universal features of a physical phenomenon need to be uncovered.

The random-matrix ensembles most frequently studied in the recent literature can be classified into three weakly overlapping categories: (i) zero-dimensional matrix field theories, which have found applications in disordered metallic grains [2–4], quantum chaos [5], nuclear physics [6], QCD [7] and two-dimensional (2D) quantum gravity [8]; (ii) Brownian motion ensembles, which have been used to describe quasi-one-dimensional disordered conductors [9–11], crossover ensembles [12], parametric correlations [13,14], and wave propagation in disordered media [15]; and (iii) one-dimensional matrix field theories, which have been applied to matrix quantum mechanics [16], $c=1$ string field theory [8], quantum chaos, and disordered metallic grains in the presence of external adiabatic perturbations [17]. The techniques developed in the literature to solve random-matrix problems vary significantly from one category to the other. The most common techniques to study zero-dimensional matrix field theories are orthogonal polynomials [1], Efetov's coset method [4,18], Brézin's method of anticommuting variables [19], the graded eigenvalue method [20], Kazakov's contour integral representation theory [21], Korteweg–de Vries (KdV) hierarchy of equations [8], topological expansions [8], and the functional derivative approach [22]. Brownian motion ensembles have been studied by means of biorthogonal functions [12], Bethe ansatz methods [23], Dyson's hydrodynamical equations [24], moment expansion techniques [10], and supersymmetry [25]. Finally, one-dimensional matrix field theories are most conveniently solved by large-N QCD techniques [27], supersymmetry [17], transfer-matrix techniques [4], and continuous Liouville theory [8].

A typical random-matrix ensemble is characterized by the probability distribution

$$P(M)=Z^{-1}\exp[-N\,\mathrm{tr}\,V(M)], \tag{1}$$

where $V(M)$ is a phenomenological confining potential, usually chosen so as to make the average level density agree with experimental observations, and N is the number of eigenvalues of M. The simplest choice is of course the harmonic potential $V(M)=k^2M^2/2$, for which the average level density reads

$$\rho(x)=\frac{Nk^2}{2\pi}\sqrt{\frac{4}{k^2}-x^2}. \tag{2}$$

This is the well-known Wigner's semicircle law. Other possibilities include cases where $V(M)$ is a polynomial of degree $2n$, which yields

$$\rho(x)=\frac{P_{2n-2}(x)}{\pi}\sqrt{a^2-x^2}, \tag{3}$$

where a (the edge of the spectrum) and the coefficients of the polynomial $P_{2n-2}(x)$ are determined by the coefficients of $V(M)$. These results can be derived by orthogonal polynomials techniques [28] and saddle point methods [29]. Observe that the nonuniversal structure of $\rho(x)$ is quite general. We would like to stress that ensembles defined by $V(M)$ being a polynomial have found important applications in nonperturbative approaches to the theory of matter coupled to 2D quantum gravity [8].

It is now rather well established in the literature that ensembles like Eq. (1) exhibit universal behavior if correlations are measured on the scale of a few mean-level spacings ($\Delta\simeq 1/N$). The universal behavior has been classified into three types: the bulk universality, characterized by a sine-kernel [1], the hard-edge universality, characterized by a Bessel kernel [30] and the soft-edge universality [31]. The terminology hard and soft edges has been introduced in Refs. [30] and [31] to distinguish between bounded and unbounded eigenvalue supports, respectively.

Recently, Brézin and Zee [26,27] (see also Ref. [32]) rediscovered a somewhat weaker universality class in this problem. They showed that when correlations are measured on scales of order $O(N^0)$, one obtains functions that depend on the confining potential only through its end points. To be specific, they proved that for ensembles defined by $V(M)$ of polynomial form, the two-point density-density correlator is given by

$$\rho(x,y)=\frac{1}{2\beta\pi^2}\frac{\partial^2}{\partial x\partial y}\ln\left|\frac{\sqrt{(b-x)(x-a)(b-y)(y-a)}-ab-xy+(a+b)(x+y)/2}{-\sqrt{(b-x)(x-a)(b-y)(y-a)}-ab-xy+(a+b)(x+y)/2}\right|, \tag{4}$$

where $a<x,y<b$ and $[a,b]$ is the support of the spectrum. This remarkable result has been shown [34] to be valid in cases where $V(M)$ is nonpolynomial and infinite (hard wall potential) at the points $x=a$ and $x=b$. Recently, generalizations of Eq. (4) to more complicated ensembles [33], such as those where the average density has support on several nonoverlapping intervals, have been derived.

In mesoscopic systems formulas of this type have shown up in the calculation of two-point functions relevant to describing universal fluctuations of transport observables. In this application of RMT the presence or the absence of a hard edge in the spectrum is crucial for determining the exact value of the amplitude of the fluctuations of the observables, such as the conductance of a two-probe device.

It is quite instructive to use Eq. (4) to recover some well-known results of RMT by taking certain limits. For instance, if we let $a\to-b$ and $b\to\infty$ we obtain the Dyson-Mehta formula

$$\rho(x,y)=\frac{1}{2\beta\pi^2}\frac{\partial^2}{\partial x\partial y}\ln|x-y|, \tag{5}$$

which describes fluctuations of linear statistics in ensembles of unbounded eigenvalue support. If we take $a\to 0$ and $b\to\infty$ we recover a formula, which has been derived originally by Beenakker [22] in the context of the global maximum-entropy approach to disordered conductors,

$$\rho(x,y)=\frac{1}{2\beta\pi^2}\frac{\partial^2}{\partial x\partial y}\ln\left|\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}\right|. \tag{6}$$

This formula was later shown to be exact for describing fluctuations of transport observables in open ballistic cavities [35]. To disordered conductors with quasi-one-dimensional geometries Eq. (6) does not apply, mainly due to the presence of diffusion modes, which give rise to additional correlations between transmission eigenvalues. In this case, the appropriate random-matrix ensemble does not have the form shown in Eq. (1), but must be generated from a Brownian-motion model, namely, the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation [9]. So far, extensions of formulas like Eq. (6) to more general geometries have not been successful within the framework of random matrix theory, although results obtained from diagrammatic calculations are available in the literature for these cases as well.

More recently, Brézin and Zee extended their result to ensembles in the category of one-dimensional matrix field theories. Using the language of large-N QCD, they showed that on scales of $O(N^0)$, the dynamical two-point density-density correlator has the form

$$\rho(x,y;u)=\frac{1}{2\beta\pi^2}\frac{\partial^2}{\partial x\partial y}\ln\left|\frac{A(+,u)}{A(-,u)}\right|, \tag{7}$$

where

$$A(\sigma,u)=\sigma\sqrt{(b-x)(x-a)(b-y)(y-a)}-ab-xy+(a+b)(x+y)/2+\frac{(b-a)^2}{2}\sinh^2 u. \tag{8}$$

In this formula the parameter u labels a continuous family of ensembles of random matrices with parametric dependence. In the language of one-dimensional matrix field theory $u=u(t)$ is a well-behaved function of the target space variable t (usually interpreted as time). The importance of this result in mesoscopic physics can be illustrated by two simple particular cases. If we take the limits $a\to-b$, $u\to X^2/(2b)$ and $b\to\infty$ we get

$$\rho(x,y;X)=\frac{1}{\beta\pi^2}\mathrm{Re}\left\{\frac{1}{(i(x-y)+X^2)^2}\right\}, \tag{9}$$

which is the result found by Altshuler [13,36] and collaborators, using diagrammatic expansions to study the effect of external parametric perturbations in the density-density correlator of disordered metallic grains. Similarly, if we let $a\to 0$, $u\to X^2/\sqrt{b}$ and $b\to\infty$ we get (after changing variables $x\to\sqrt{\mu}$ and $y\to\sqrt{\nu}$) the expression

$$\tilde{\rho}(\mu,\nu;X)=\frac{1}{\beta\pi^2}\sum_{\sigma=\pm1}\mathrm{Re}\left\{\frac{1}{(i(\mu+\sigma\nu)+X^2)^2}\right\}, \tag{10}$$

which is precisely the result found in Ref. [37] for the parametric density-density correlator of transmission eigenvalues in open ballistic cavities.

In Ref. [27] it has been shown that Eq. (7) is valid for confining potentials, $V(M)$, which are polynomials. Unfortunately, the explicit calculations of this work are somewhat cumbersome and extensions to more general functional forms of $V(M)$ seem unlikely. However, we know from Ref. [34] that the static correlator $(u=0)$ is valid under quite weak assumptions about the form of the confining potential $V(M)$. The natural question seems to be to what extent the dynamical correlator $(u\neq 0)$ is independent of the functional form of $V(M)$.

In this paper we provide a first step towards answering this question. We give an alternative derivation of Eq. (7) that has the appealing feature of being much simpler but also more general, since our strongest assumption about $V(M)$ is that it is capable of providing confinement of the levels within a single interval. In Sec. II, we introduce the S-matrix Brownian motion ensemble, which provides level confinement in a natural way by means of symmetry requirements. We sketch briefly how the Fokker-Planck equation is obtained from the stochastic process in the S-matrix manifold. In Sec. III, we show that Eq. (7) is a direct consequence of Dyson's hydrodynamical approximation to the dynamics of an S matrix undergoing Brownian motion on its manifold. A summary and conclusions are given in Sec. IV.

II. THE S-MATRIX BROWNIAN MOTION ENSEMBLE

In the early days of RMT, Brownian motion ensembles were introduced by Dyson [24] with the motivation of improving the physical content of the stationary theory of Wigner. While quite successful as a mathematical tool for describing local statistics of the spectra of complex nuclei, the original Wigner ensembles were known to make quite wrong predictions for global properties, such as the average level density. In a remarkable paper [24], Dyson showed how to build a new class of random-matrix ensembles with a lot of desirable properties and, in this way, he laid the foundations of modern nonequilibrium RMT. The ensemble introduced by Dyson is physically reasonable, mathematically tractable, recovers Wigner's local statistics, and agrees well with the observed eigenvalue density. Inspired by Brownian motion theory, Dyson made the following assumptions: (1) the true Hamiltonian can be derived from the bare one by equally likely random perturbations; (2) the strength of the perturbation is controlled by a parameter whose physical meaning depends on the model system; (3) the stochastic process is biased by fictitious driving forces such that it produces the observed eigenvalue density. Dyson's Brownian motion model has since been successfully applied to a large variety of physical systems, including those with partially broken symmetries and/or adiabatic parametric perturbations. Recent improvements of Dyson's theory consisted of generalizations of his third hypothesis to include cases in which the symmetries of the physical system impose a particular biasing of the stochastic process, which cannot be mimicked by fictitious forces. Examples of such Brownian motion models are the DMPK equation, the Ω-matrix ensemble, and the S-matrix ensemble.

The S matrix by definition establishes a relation between incoming flux amplitudes I and I' and outgoing ones O and O':

$$S\begin{pmatrix} I \\ I' \end{pmatrix} = \begin{pmatrix} O \\ O' \end{pmatrix}. \tag{11}$$

Convervation of probability implies that S is unitary. Mathematically, unitary matrices can be parametrized by the following polar decomposition [35]:

$$S=\begin{pmatrix} u^{(1)} & 0 \\ 0 & u^{(2)} \end{pmatrix}\begin{pmatrix} -\sqrt{1-\tau} & \sqrt{\tau} \\ \sqrt{\tau} & \sqrt{1-\tau} \end{pmatrix}\begin{pmatrix} u^{(3)} & 0 \\ 0 & u^{(4)} \end{pmatrix}, \tag{12}$$

where $u^{(i)}(i=1,\ldots,4)$ are $N\times N$ unitary matrices and τ denotes an $N\times N$ diagonal matrix with real eigenvalues $0\leqslant\tau_\alpha\leqslant 1$ $(\alpha=1,2,\ldots,N)$.

The stationary (maximum entropy) S-matrix ensemble has been introduced in Ref. [35], and the Brownian motion ensemble in Refs. [12] and [38]. While the maximum entropy ensemble is somewhat straightforward (the S matrix simply covers its manifold ergodically), the Brownian motion ensemble has some subtlety. Following the generalized form of Dyson's hypothesis, the ensemble can be constructed by a stochastic process, which has built into it the symmetries of the problem.

So, let $S(t)$ denote a point in the S-matrix manifold, and let $S_0(\delta t)$ be a random unitary matrix (taken from a uniform distribution), where the parameter t is phenomenological in Dyson's sense. The point $S(t)$ performs isotropic Brownian motion determined by the law

$$S(t+\delta t)=S(t)S_0(\delta t). \tag{13}$$

We choose the functional dependence of $S_0(\delta t)$ such that $S_0(\delta t\to 0)\to 1$. Without going into the details of the derivation, which can be found in Ref. [12], the stochastic process above can be described in the continuum limit by the Fokker-Planck equation

$$\frac{\partial W}{\partial t}=\sum_{i=1}^{N}\left(-\frac{\partial}{\partial \tau_i}\mathcal{D}_i^{(1)}+\frac{\partial^2}{\partial \tau_i^2}\mathcal{D}_i^{(2)}\right)W, \tag{14}$$

where the drift and diffusion coefficients are given, respectively, by

$$\mathcal{D}_i^{(1)}=-(1+\beta/2)\tau_i+\beta/2+\beta\sum_{j(\neq i)}\frac{\tau_i(1-\tau_i)}{\tau_i-\tau_j} \tag{15}$$

$$\mathcal{D}_i^{(2)}=\tau_i(1-\tau_i). \tag{16}$$

The parameter β labels symmetry classes: $\beta=1$ for orthogonal systems, whereas $\beta=2$ and $\beta=4$ for unitary and symplectic systems, respectively. Observe that in the limit $t\to\infty$ Eq. (14) has a maximum-entropy stationary solution

$$W_{\mathrm{eq}}(\tau)=Z^{-1}\prod_{i<j}|\tau_i-\tau_j|^{\beta}\prod_{i=1}^{N}\tau_i^{\beta/2-1}, \tag{17}$$

in which Z is a normalization constant. Note the presence of the typical random-matrix eigenvalue repulsion since $W_{\mathrm{eq}}\to 0$ if any $\tau_i\to\tau_j$. Equation (14) defines the S-matrix Brownian motion ensemble.

III. BRÉZIN-ZEE DYNAMICAL CORRELATOR

In Ref. [27] the following one-dimensional matrix field theory has been considered

$$P(M)=Z^{-1}\exp-\int_{-\infty}^{\infty}dt\ \mathrm{Tr}[K(d/dt)M+V(M)],\qquad(18)$$

which corresponds to a $c=1$ string field theory. Using large-N QCD techniques with the generalized gluon propagator

$$D_{ij,kl}(t)\equiv\langle M_{ij}(t)M_{kl}(0)\rangle\propto\exp[-u(t)],\qquad(19)$$

in which $u(t)$ is a smooth function with finite value at $t=0$, they showed that the two-point dynamical density-density correlator has the form shown in Eq. (7).

In this section we demonstrate how this result can be obtained from the S-matrix Brownian motion ensemble. The crucial point is that this ensemble has a natural confinement of levels induced by the symmetry of the unitary group. So, if Eq. (7) is as generally valid as its static counterpart, then one might expect that it may be obtained from standard nonequilibrium RMT, provided one finds the appropriate dynamical biasing of the stochastic process. Here, we assume that the symmetries of the unitary group provide the dominant mechanism for the relaxation of the correlators in the universal domain, which is, as discussed in the Introduction, of $O(N^0)$.

We start by writing Eq. (14) in the form

$$\frac{\partial P}{\partial t}=\sum_{i=1}^{N}\left[-\frac{\partial}{\partial\varphi_i}D_i+\frac{1}{4}\frac{\partial^2}{\partial\varphi_i^2}\right]P,\qquad(20)$$

where $\varphi_i=\arcsin\sqrt{\tau_i}$ and

$$D_i=\frac{\beta-2}{4}\csc(2\varphi_i)+\frac{\beta}{4}\cot(2\varphi_i)-\frac{\beta}{2}\sum_{j(\neq i)}\frac{\sin(2\varphi_i)}{\cos(2\varphi_i)-\cos(2\varphi_j)}.\qquad(21)$$

Define the average level density

$$\sigma(\varphi,t)\equiv\left\langle\sum_{n=1}^{N}\delta(\varphi-\varphi_n)\right\rangle_t,\qquad(22)$$

which has the useful property $\sigma(\varphi,\infty)=2N/\pi\equiv\sigma_0$. Note that the choice of these variables is motivated by the technical need to unfold the spectrum. Nonetheless, the need for such a procedure does not impose severe restrictions on the functional form of the one-body potential $V(\tau)$. The hydrodynamical equations that we shall introduce briefly, require for their solution unfolding of the spectrum on an interval of $O(N^0)$, which corresponds to the universal regime of interest. Close to the end points the potential is still free of constraints.

Following Dyson, we now take the hydrodynamic limit of Eq. (20). The technical details of such a standard procedure are described in detail in Ref. [24], (see also Ref. [14]). The final result is that the average level density in the large-N limit obeys the nonlinear equation

$$\frac{\partial\sigma(\varphi,t)}{\partial t}=-\beta\frac{\partial}{\partial\varphi}\left(\sigma(\varphi,t)\frac{\partial}{\partial\varphi}\wp\int_0^{\pi/2}\sigma(\varphi',t)\ln\left|\cos(2\varphi)-\cos(2\varphi')\right|d\varphi'\right).\qquad(23)$$

For our purpose it is sufficient to solve this equation by linearizing it around the equilibrium solution. We define the function

$$\delta\sigma(\varphi,t)\equiv\sigma(\varphi,t)-\sigma_0,\qquad(24)$$

which, by virtue of Eq. (23), satisfies the linear equation

$$\frac{\partial\delta\sigma(\varphi,t)}{\partial t}=-\beta\sigma_0\wp\int_0^{\pi/2}\delta\sigma(\varphi',t)Q(\varphi,\varphi')d\varphi',\qquad(25)$$

where

$$Q(\varphi,\varphi')=4\sum_{n=1}^{\infty}n\cos(2n\varphi)\cos(2n\varphi').\qquad(26)$$

This equation in the main result of this section and some comments are in order. It is for our purpose more important than the Brownian-motion Fokker-Planck equation itself, because it is stripped of nonuniversal features. The form of the kernel $Q(\varphi,\varphi')$ is fixed by the equilibrium level repulsion potential, which yields (as we discussed in the Introduction), under quite weak assumptions, the universal static density-density correlator [see Eq. (4)]. The effect of the drift and diffusion coefficients of the Brownian motion ensemble on this equation is twofold: first it provides the bounds for the integral, which in fact is fixed by symmetry, and second it must be such that the average eigenvalue density is roughly constant on a scale of $O(N^0)$. This last constraint comes also naturally from the structure of the S-matrix manifold. This integrodifferential equation can be solved exactly by Fourier series. With this in mind we propose the expansion

$$\delta\sigma(\varphi,t)=\sum_{n=1}^{\infty}\delta\sigma_n(t)\cos(2n\varphi),\qquad(27)$$

and after substituting Eq. (27) into Eq. (25) we find the ordinary differential equation

$$\frac{d\,\delta\sigma_n(t)}{dt}=-\beta\pi n\sigma_0\delta\sigma_n(t),\qquad(28)$$

which can be easily solved to yield

$$\delta\sigma_n(t)=\delta\sigma_n(0)e^{-nu(t)},\qquad(29)$$

where $u(t)=\beta\pi\sigma_0 t$. We are now in a position to calculate the time-dependent two-point correlator, which is defined as the following equilibrium average

$$S(\varphi,\varphi';t)\equiv\langle\delta\sigma(\varphi,t)\delta\sigma(\varphi,0)\rangle_{\rm eq}$$
$$\equiv\sum_{n=1}^{\infty}\langle\delta\sigma_n(t)\delta\sigma_n(0)\rangle_{\rm eq}\cos(2n\varphi)\cos(2n\varphi'). \tag{30}$$

Using Eq. (29) we find

$$\langle\delta\sigma_n(t)\delta\sigma_n(0)\rangle_{\rm eq}=e^{-nu(t)}\langle\delta\sigma_n(0)\delta\sigma_n(0)\rangle_{\rm eq}$$
$$=e^{-nu(t)}\frac{8n}{\beta\pi^2}. \tag{31}$$

In Eq. (31) we have inserted the value of the static correlator, which can be calculated directly from the equilibrium solution by standard methods. Inserting (31) into (30) and evaluating the infinite sum we find the closed-form result

$$S(\varphi,\varphi';u)=\frac{1}{2\beta\pi^2}\frac{\partial}{\partial\varphi\partial\varphi'}\ln\left|\frac{\cosh u-\cos2(\varphi+\varphi')}{\cosh u-\cos2(\varphi-\varphi')}\right|. \tag{32}$$

To make contact with the Brézin-Zee dynamical correlator we need to perform the following change of variables:

$$x=\frac{b+a}{2}-\frac{b-a}{2}\cos2\varphi,$$
$$y=\frac{b+a}{2}-\frac{b-a}{2}\cos2\varphi', \tag{33}$$

after which we find

$$\rho(x,y;u)=\frac{1}{2\beta\pi^2}\frac{\partial^2}{\partial x\partial y}\ln\left|\frac{A(+,u)}{A(-,u)}\right|, \tag{34}$$

where

$$A(\sigma,u)=\sigma\sqrt{(b-x)(x-a)(b-y)(y-a)}-ab-xy$$
$$+(a+b)(x+y)/2+\frac{(b-a)^2}{2}\sinh^2u. \tag{35}$$

This is precisely the result obtained in Ref. [27].

IV. SUMMARY AND CONCLUSIONS

In this work we have demonstrated how Brézin-Zee's dynamical correlator can be obtained from an S-matrix Brownian motion model. A clear advantage of our approach is its simplicity, in comparison with the diagrammatic method of Ref. [27]. Furthermore, our ensemble of matrices has natural confinement of levels, which is induced by symmetry. In this way, the fundamental role of the potential $V(M)$ (see Introduction) for providing the spectral end-point dependence of the universal correlators is incorporated into the theory from the beginning, without any particular assumption on the functional form of $V(M)$. We believe that such an approach has considerable practical importance, particularly in understanding the connections between the various classes of random-matrix ensembles and the related model systems, for that matter. Extensions of our results to more elaborate cases, such as ensembles where the eigenvalue support contains multiple nonoverlapping intervals and crossover problems does seem possible, but will be left to the future.

ACKNOWLEDGMENTS

This work was partially supported by CAPES, CNPq, and Facepe (Brazilian Agencies).

[1] M. L. Mehta, *Random Matrices*, 2nd ed. (Academic, New York, 1991).
[2] L. P. Gorkov and G. M. Eliashberg, Zh. Éksp. Teor. Fiz. **48**, 1407 (1965) [Sov. Phys. JETP **21**, 940 (1965)].
[3] B. L. Altshuler and B. I. Shklovskii, Zh. Éksp. Teor. Fiz. **91**, 220 (1986) [Sov. Phys. JETP **64**, 127 (1986)].
[4] K. B. Efetov, Adv. Phys. **32**, 53 (1983).
[5] See, e.g., F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 1992).
[6] See, e.g., *Statistical Theory of Spectra: Fluctuations*, edited by C. E. Porter (Academic, New York, 1965).
[7] J. J. M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70**, 3852 (1993).
[8] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, Phys. Rep. **254**, 1 (1995).
[9] O. N. Dorokhov, Zh. Éksp. Teor. Fiz. **85**, 1040 (1983) [Sov. Phys. JETP **58**, 606 (1983)]; P. A. Mello, P. Pereyra, and N. Kumar, Ann. Phys. (N.Y.) **181**, 290 (1988).
[10] P. A. Mello and A. D. Stone, Phys. Rev. B **44**, 3559 (1991).
[11] A. M. S. Macêdo and J. T. Chalker, Phys. Rev. B **46**, 14 985 (1992).
[12] K. Frahm and J.-L. Pichard J. Phys. (France) I **5**, 877 (1995).
[13] B. D. Simons, P. A. Lee, and B. L. Altshuler, Phys. Rev. Lett. **70**, 4122 (1993).
[14] C. W. J. Beenakker, Phys. Rev. Lett. **70**, 4126 (1993); C. W. J. Beenakker and B. Rejaei, Physica A **203**, 61 (1994).
[15] See, e.g., C. W. J. Beenakker, J. C. J. Paasschens, and P. W. Brouwer, Phys. Rev. Lett. **76**, 1368 (1996).
[16] See, e.g., C. Itzykson and J. M. Drouffe, *Statistical Field Theory* (Cambridge University Press, Cambridge, 1991), Vol. 2.
[17] B. D. Simons, P. A. Lee, and B. L. Altshuler, Nucl. Phys. B **409**, 487 (1993); Phys. Rev. Lett. **72**, 64 (1994).
[18] J. J. M. Verbaarshot, H. A. Weidenmüller, and M. R. Zirnbauer, Phys. Rep. **129**, 367 (1985).
[19] E. Brézin, in *Lecture Notes in Physics*, Vol. 216 (Springer-Verlag, Berlin, 1985), p. 115.
[20] T. Guhr, J. Math. Phys. **32**, 336 (1991).
[21] V. A. Kazakov, Nucl. Phys. B **354**, 614 (1991).
[22] C. W. J. Beenakker, Phys. Rev. Lett. **70**, 1155 (1993); Phys. Rev. B **47**, 15 763 (1993).
[23] B. Sutherland, J. Math. Phys. **12**, 251 (1971).
[24] F. J. Dyson, J. Math. Phys. **13**, 90 (1972).
[25] S. Iida, H. A. Weidenmüller, and J. A. Zuk, Ann. Phys. (N.Y.) **200**, 219 (1990).

[26] E. Brézin and A. Zee, Nucl. Phys. B **402**, 613 (1993).
[27] E. Brézin and A. Zee, Phys. Rev. E **49**, 2588 (1994).
[28] D. Bessis, C. Itzykson, and J.-B. Zuber, Adv. Appl. Math. **1**, 109 (1980).
[29] E. Brézin, C. Itzykson, G. Parisi, and J.-B. Zuber, Commun. Math. Phys. **59**, 35 (1978).
[30] T. Nagao and M. Wadati, J. Phys. Soc. Jpn. **60**, 3298 (1991); C. A. Tracy and H. Widom, Commun. Math. Phys. **161**, 289 (1994).
[31] C. A. Tracy and H. Widom, Commun. Math. Phys. **159**, 151 (1994).
[32] J. Ambjørn, J. Jurkiewicz, and Y. Makeenko, Phys. Lett. B **251**, 517 (1990).
[33] J. Ambjørn and G. Akemann (unpublished).
[34] C. W. J. Beenakker, Nucl. Phys. B **422**, 515 (1994).
[35] H. U. Baranger and P. A. Mello, Phys. Rev. Lett. **73**, 142 (1994); R. A. Jalabert, J.-L. Pichard, and C. W. J. Beenakker, Europhys. Lett. **27**, 255 (1994).
[36] A. Szafer and B. L. Altshuler, Phys. Rev. Lett. **70**, 587 (1993).
[37] A. M. S. Macêdo, Phys. Rev. B **53**, 8411 (1996).
[38] J. Rau, Phys. Rev. B **51**, 7734 (1995).